Properties and applications of copulas: A brief survey

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1 Introduction

A *copula* is a function which joins or "couples" a multivariate distribution function to its one-dimensional marginal distribution functions. The word "copula" was first used in a mathematical or statistical sense by Sklar (1959) in the theorem which bears his name (see the next section). But the functions themselves predate the use of the term, appearing in the work of Hoeffding, Fréchet, Dall'Aglio, and many others. Over the past forty years or so, copulas have played an important role in several areas of statistics. As Fisher (1997) notes in the *Encyclopedia of Statistical Sciences*, "Copulas [are] of interest to statisticians for two main reasons: First, as a way of studying scale-free measures of dependence; and secondly, as a starting point for constructing families of bivariate distributions, ..." In Sections 2 through 5 we present the basic properties of copulas and several families of copulas useful in statistical modeling, and in Sections 6 and 7 we explore the relationships between copulas and dependence properties and measures of association. The concept of a quasi-copula was introduced by Alsina et al. (1993) in order to characterize operations on distribution functions that can or cannot be derived from operations on random variables. We discuss quasi-copulas and their relationship with copulas in Sections 8 and 9. We conclude with extensions to higher dimensions in Section 10, and a few open problems.

This brief survey is necessarily incomplete. Readers seeking to learn more about copulas and quasi-copulas will find the monographs by Hutchinson and Lai (1990), Joe (1997), and Nelsen (1999) and conference proceedings edited by Beneš and Štěpán (1997), Cuadras et al. (2002), Dall'Aglio et al. (1991), and Rüschendorf et al. (1996) useful.

2 Copulas

Copulas can be defined informally as follows: Let X and Y be continuous random variables with distribution functions $F(x) = P(X \le x)$ and $G(y) = P(Y \le y)$, and joint distribution function $H(x, y) = P(X \le x, Y \le y)$. For every (x, y) in $[-\infty, \infty]^2$ consider the point in $\mathbf{I}^3$ ($\mathbf{I} = [0, 1]$) with coordinates $(F(x), G(y), H(x, y))$. This mapping from $\mathbf{I}^2$ to $\mathbf{I}$ is a *copula*. Copulas are also known as *dependence functions or uniform representations.*

Formally we have

DEFINITION 2.1. A (two-dimensional)*copula* is a function $C : \mathbf{I}^2 \to \mathbf{I}$ such that

(C1) $C(0,x) = C(x,0) = 0$ and $C(1,x) = C(x,1) = x$ for all $x \in \mathbf{I}$;

(C2) C is 2-*increasing*: for $a,b,c,d \in \mathbf{I}$ with $a \le b$ and $c \le d$,

$$V_C\Big([a,b] \times [c,d]\Big) = C(b,d) - C(a,d) - C(b,c) + C(a,c) \ge 0.$$

The function V_C in (C2) is called the *C-volume* of the rectangle $[a,b] \times [c,d]$. Equivalently, a copula is the restriction to the unit square $\mathbf{I}^2$ of a bivariate distribution function whose margins are uniform on $\mathbf{I}$. Note that a copula C induces a probability measure on $\mathbf{I}^2$ via $V_C\Big([0,u] \times [0,v]\Big) = C(u,v)$.

It is easy to see that function $\Pi(u,v) = uv$ satisfies conditions (C1) and (C2), and hence is a copula. Note that the Π-volume V_Π of a rectangle is its area. The copula Π, called the *product copula*, has an important statistical interpretation (see below).

The informal and formal definitions are connected by the following theorem (Sklar, 1959), which also partially explains the importance of copulas in statistical modeling.

SKLAR'S THEOREM: *Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then there exists a copula C such that H(x,y) = C(F(x),G(y)). Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with marginals F and G. Furthermore, if F and G are continuous, C is unique.*

It is easy to show that, as a consequence of the 2-increasing property (C2) in Definition 2.1, for any copula C we have

(C3) C is nondecreasing in each variable, and

(C4) C satisfies the followining Lipschitz condition: for every a,b,c,d in $\mathbf{I}$,

$$|C(b,d) - C(a,c)| \le |b-a| + |d-c|\,.$$

Consequently copulas are uniformly continuous. However, properties (C3) and (C4) together are not equivalent to (C2) (see Section 8 below).

Given a joint distribution function H with continuous marginals F and G, as in Sklars Theorem, it is easy to construct the corresponding copula: $C(u,v) = H(F^{(-1)}(u), G^{(-1)}(v))$, where $F^{(-1)}$ is the *cadlag inverse* of F, given by $F^{(-1)}(u) = \sup\{x\,|F(x) \le u\}$ (and similarly for $G^{(-1)}$). Note as well that if X and Y are continuous random variables with distribution functions as above, then C is the joint distribution function for the random variables $U = F(X)$ and $V = G(Y)$ (recall that $F(X)$ and $G(Y)$ are uniformly distributed on $\mathbf{I}$).

When a copula C, considered as a joint distribution function on $\mathbf{I}^2$, possesses a joint density $\partial^2 C(u,v)/\partial u \partial v$, then C is *absolutely continuous*. Otherwise, C may be *singular*, or possess both an absolutely continuous component and a singular component.

It is an elementary exercise to show that if H is a bivariate distribution function with marginals F and G, then

$$\max\{F(x)+G(y)-1,0\} \le H(x,y) \le \min\{F(x),G(y)\} \tag{1}$$

or [since $H(x,y) = C(F(x),G(y))$]

$$W(u,v) = \max\{u+v-1,0\} \le C(u,v) \le \min\{u,v\} = M(u,v). \tag{2}$$

This inequality is known as the *Fréchet-Hoeffding bounds inequality*, and the functions W and M as the Fréchet-Hoeffding lower and upper bounds, respectively, in recognition of the pioneering work in this field by Hoeffding (1940, 1941) and Fréchet (1951). Furthermore, M and W are themselves copulas. Hence the graph of any copula is a continuous surface within the unit cube $\mathbf{I}^3$ whose boundary is the skew quadrilateral with vertices $(0,0,0),(1,0,0),(0,1,0)$, and $(1,1,1)$; and this graph lies between the graphs of the Fréchet-Hoeffding bounds, i.e., the surfaces $z = W(u,v)$ and $z = M(u,v)$.

The copulas M, W, and Π have important statistical interpretations. Let X and Y be continuous random variables, then:

(i) the copula of X and Y is $M(u,v)$ if and only if each of X and Y is almost surely an increasing function of the other;

(ii) the copula of X and Y is $W(u,v)$ if and only if each of X and Y is almost surely a decreasing function of the other;

(iii) the copula of X and Y is $\Pi(u,v) = uv$ if and only if X and Y are independent.

A note on notation: We will write $C_{X,Y}$ for "the copula of X and Y is C," i.e., when the identification of the copula with the random variables is advantageous. With this notation, part (iii) of the above remark could be rephrased as: X and Y are independent if and only if $C_{X,Y} = \Pi$.

If α, β are almost surely increasing functions of X, Y respectively, then the copula of $\alpha(X)$ and $\beta(Y)$ is the same as the copula of X and Y—i.e., $C_{\alpha(X),\beta(Y)} = C_{X,Y}$— hence it is the copula which captures the "nonparametric," "distribution-free" or "scale-invariant" nature of the dependence between X and Y. When at least one of α and β is strictly decreasing, the copula changes in a predictable way:

(i) If α is strictly increasing and β is strictly decreasing, then

$$C_{\alpha(X),\beta(Y)}(u,v) = u - C_{X,Y}(u,1-v);$$

(ii) If α is strictly decreasing and β is strictly increasing, then

$$C_{\alpha(X),\beta(Y)}(u,v) = v - C_{X,Y}(1-u,v);$$

(iii) If α and β are both strictly decreasing, then

$$C_{\alpha(X),\beta(Y)}(u,v) = u+v-1+C_{X,Y}(1-u,1-v).$$

3 Families of Copulas

If one has a collection of copulas, then using Sklar's theorem, one can construct, bivariate distributions with arbitrary margins. Thus, for the purposes of statistical modeling, it is desirable to have a collection of copulas at ones disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameters (members of such families are often denoted C_θ, $C_{\alpha,\beta}$, etc.). We now present a very brief overview of some parametric families of copulas. Extensive surveys of families of copulas can be found in Hutchinson and Lai (1990), Joe (1997), and Nelsen (1999).

3.1 The Farlie-Gumbel-Morgenstern family

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \theta \in [-1,1] .$$

These are the only copulas whose functional form is a polynomial quadratic in u and in v. They are commonly denoted FGM copulas.

Members of the FGM family are *symmetric*, i.e., $C_\theta(u,v) = C_\theta(v,u)$ for all (u,v) in $\mathbf{I}^2$. A pair (X,Y) of random variables is said to be *exchangeable* if the vectors (X,Y) and (Y,X) are identically distributed. For identically distributed continuous random variables, exchangeability is equivalent to the symmetry of the copula.

3.2 Copulas cubic in u and in v

$$C(u,v) \ = \ uv + uv(1-u)(1-v)[\alpha uv + \beta u(1-v) + \gamma v(1-u) + \delta(1-u)(1-v)],$$

where $\alpha, \beta, \gamma, \delta$ are real constants chosen so that the points $(\alpha,\beta), (\alpha,\gamma), (\delta,\beta)$, and (δ,γ) all lie in the set $[-1,2] \times [-2,1] \cup \{(x,y)\,|x^2 - xy + y^2 - 3x + 3y \le 0\}$. When $\alpha = \beta = \gamma = \delta = \theta$, C is quadratic rather than cubic in u and in v, and the copulas are members of the FGM family. Unlike FGM copulas, copulas cubic in u and in v may be asymmetric. For further details, see Nelsen et al. (1997) and Nelsen (1999).

3.3 Normal copulas

Let $N_\rho(x,y)$ denote the standard bivariate normal joint distribution function with correlation coefficient ρ. Then C_ρ, the copula corresponding to N_ρ, is given by $C_\rho(u,v) = N_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$ [where Φ denotes the standard normal distribution function]. Since there is no closed form expression for Φ^{-1}, there is no closed form expression for N_ρ. However, N_ρ can be evaluated approximately in order to construct bivariate distribution functions with the same dependence structure as the standard bivariate normal distribution function but with non-normal marginals.

DEFINITION 3.1. For a pair X, Y of random variables with marginal distribution functions F, G, respectively, and joint distribution function H, the marginal survival functions $\overline{F}, \overline{G}$, and joint survival function $\overline{H}$ are given by $\overline{F}(x) = P\,[X > x]\,, \overline{G}(y) = P\,[Y > y]$ and $\overline{H}(x,y) = P\,[X > x, Y > y]$ respectively. The function $\hat{C}$ which couples the joint survival function to its marginal survival functions is called a *survival copula*: $\overline{H}(x,y) = \hat{C}(\overline{F}(x), \overline{G}(y))$.

It is easy to show that $\hat{C}$ is a copula, and is related to the (ordinary) copula C of X and Y via the equation $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. See Nelsen (1999) for details.

3.4 Cuadras-Augé copulas

$$C_{\alpha,\beta}(u,v) \;= \min(u^{1-\alpha}v, uv^{1-\beta}), \quad \alpha, \beta \in [0,1].$$

The case $\alpha = \beta$ appeared first in Cuadras and Augé (1981), and they are the survival copulas associated with the Marshall and Olkin (1967) bivariate exponential distribution. Note that $C_{\alpha,0} = C_{0,\beta} = \Pi$ and $C_{1,1} = M$. When $\alpha, \beta \in (0,1)$, $C_{\alpha,\beta}$ possesses both an absolutely continuous com-ponent and a singular component.

4 Archimedean copulas

Let ϕ be a continuous strictly decreasing function from $\mathbf{I}$ to $[0,\infty]$ such that $\phi(1) = 0$, and let $\phi^{[-1]}$ denote the "pseudo-inverse" of $\phi : \phi^{[-1]}(t) = \phi^{-1}(t)$ for $t \in [0, \phi(0)]$, and $\phi^{[-1]}(t) = 0$ for $t \geq \phi(0)$. Then $C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v))$ satisfies condition (C1) for copulas. If, in addition, ϕ is convex, then it can be shown [Schweizer and Sklar, 1983, Nelsen, 1999] that C also satisfies the 2-increasing condition (C2), and is thus a copula. Such copulas are called *Archimedean*. When $\phi(0) = \infty$, we say that C is *strict*, and when $\phi(0) < \infty$, we say that C is *non-strict*. When C is strict, $C(u,v) > 0$ for all (u,v) in $(0,1]^2$.

Here is a short list of some generators and the names associated with the copula in the literature. The parameter interval gives the values of θ for which the generator ϕ is convex (limits may be required at some values in the interval). See Nelsen (1999) for further examples.

generator	$\theta \in$	copula
$\phi(t) = (t^{-\theta} - 1)/\theta$	$[-1,\infty)$	*Clayton*
$\phi(t) = \ln\Big[\Big(1 - \theta(1-t)\Big)/t\Big]$;	$[-1,1)$	*Ali-Mikhail-Haq*
$\phi(t) = (-\ln t)^{\theta}$;	$[1,\infty)$	*Gumbel-Hougaard*
$\phi(t) = -\ln\Big[(e^{-\theta t} - 1)/(e^{-\theta} - 1)\Big]$	$[-\infty,\infty)$	*Frank*
etc.		

Archimedean copulas are widely used in applications (especially in finance, insurance, etc.) due to their simple form and nice properties (for example, most but not all extend to higher dimensions via the associativity property, see Section 9 below). Procedures exist for choosing a particular member of a given family of Archimedean copulas to fit a data set (Genest and Rivest, 1993, Wang and Wells, 2000). However, there does not seem to be a natural statistical property for random variables with an associative copula.

We now illustrate the procedure for finding a copula in a simple statistical setting. Let $\{X_1, X_2, \cdots, X_n\}$ be a set of independent and identically distributed continuous random variables with distribution function F, and let $X_{(1)} = \min\{X_1, X_2, \cdots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \cdots, X_n\}$. We now find the copula $C_{1,n}$ of $X_{(1)}$ and $X_{(n)}$.

The distribution functions F_n of $X_{(n)}$ and F_1 of $X_{(1)}$ are given by $F_n(x) = [F(x)]^n$ and $F_1(x) = 1 - [1 - F(x)]^n$. For convenience, we will first find the joint distribution function H^* and copula C^* of $-X_{(1)}$ and $X_{(n)}$, rather than $X_{(1)}$ and $X_{(n)}$:

$$\begin{aligned} H^*(s,t) &= P[-X_{(1)} \le s, X(n) \le t] \\ &= P[-s \le X_{(1)}, X_{(n)} \le t] \\ &= P[\text{ all } X_i \text{ in } [-s,t]] \\ &= \begin{cases} [F(t) - F(-s)]^n, & -s \le t, \\ 0, & -s > t, \end{cases} \\ &= [\max(F(t) - F(-s), 0)]^n. \end{aligned}$$

To obtain the copula C^*, we invert, that is, $C^*(u,v) = H^*(G^{(-1)}(u), F_n^{(-1)}(v))$, where G now denotes the distribution function of$-X_{(1)}$, $G(x) = [1 - F(-x)]^n$. Let $u = [1 - F(-s)]^n$ and $v = [F(t)]^n$, so that $F(-s) = 1 - u^{1/n}$ and $F(t) = v^{1/n}$. Thus $C^*(u,v) = \left[\max\left(u^{1/n} + v^{1/n} - 1, 0\right)\right]^n$, a member of the Clayton family of Archimedean copulas.

Now, if $C_{X,Y}$ denotes the copula of X and Y and $C_{-X,Y}$ the copula of $-X$ and Y, then $C_{X,Y}(u,v) = v - C_{-X,Y}(1-u,v)$. Thus

$$\begin{aligned} C_{1,n}(u,v) &= v - C^*(1-u,v) \\ &= v - \left[\max\left((1-u)^{1/n} + v^{1/n} - 1, 0 \max\right)\right]^n. \end{aligned}$$

Although $X_{(1)}$ and $X_{(n)}$ are clearly not independent ($C_{1,n} \neq \Pi$), they are asymptotically independent since $\lim_{n\to\infty} C_{1,n} = \Pi$.

5 Shuffles of M

Another important family of copulas (for theoretical purposes) are the *Shuffles of M*. Explicit expression for shuffles can be readily obtained, but they are often unwieldy. However we have the following informal description of the mass distribution for random variables whose copula is a shuffle (Mikusiński et al., 1992):"The mass distribution for a shuffle of M can be obtained by (2) placing the mass for M on $\mathbf{I}^2$, (3) cutting $\mathbf{I}^2$ vertically into a finite number of strips, (4) shuffling the strips with perhaps some of them flipped around their vertical axes of symmetry, and then (5) reassembling them to form the square again. The resulting mass distribution corresponds to a copula called a shuffle of M."

Random variables whose copulas are shuffles have the following statistical interpretation: If X and Y are continuous random variables whose copula is a shuffle of M, then X and Y are *mutually completely dependent* since the support is the graph of a one-to-one function. In a sense, mutual complete dependence is the "opposite" of independence. [Note: not all mutually completely dependent random variables have shuffles for their copulas.]

THEOREM 5.1 (Mikusiński et al.,1991). *For any $\epsilon > 0$, there exists a shuffle of M, which we denote C_ϵ, such that*

$$\sup_{u,v\in I} |C_\varepsilon(u,v) - \Pi(u,v)| < \varepsilon.$$

This result implies that in practice the behavior of any pair of independent random variables can be approximated so closely by a pair of mutually completely dependent random variables that it would be impossible, experimentally, to distinguish one pair from the other. But more is true—the copula Π in the theorem can be replaced by *any copula whatsoever*. In other words, the set of shuffles is *dense* (with respect to the sup norm) in the set of copulas. See Mikusiński et al. (1991) for further details. We will encounter shuffles of M again in Section 7.

6 Descriptions of dependence and measures of association

There are a variety of ways to describe and measure the dependence or association between random variables. As we noted earlier, it is the copula which captures the "nonparametric," "distribution-free" or "scale-invariant" nature of the association between random variables. Thus the focus of this section is an exploration of the role copulas play in the study of association.

6.1 Concordance

The most widely known scale-invariant measures of association are the population versions of Kendall's tau and Spearman's rho. Both measure a form of dependence known as concordance.

DEFINITION 6.1. Two observations (x_1, y_1) and (x_2, y_2) of a pair (X, Y) of continuous random variables are *concordant* if $x_1 > x_2$ and $y_1 > y_2$ or if $x_1 < x_2$ and $y_1 < y_2$, i.e., if $(x_1 - x_2)(y_1 - y_2) > 0$; and *discordant* if $x_1 > x_2$ and $y_1 < y_2$ or if $x_1 < x_2$ and $y_1 > y_2$, i.e., if $(x_1 - x_2)(y_1 - y_2) < 0$. Geometrically, two distinct points (x_1, y_1) and (x_2, y_2) in the plane are concordant if the line segment connecting them has positive slope, and discordant if the line segment has negative slope.

The sample version of the measure of association known as Kendall's tau is defined in terms of concordance as follows [Kruskal, 1958]: Let $\{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$ denote a random sample of n observations from a vector (X, Y) of continuous random variables. There are $\binom{n}{2}$ distinct pairs (x_i, y_i), (x_j, y_j) of observations each pair is either concordant or discordant. Kendall's tau is given by

$$\frac{\text{(number of concordant pairs)–(number of discordant pairs)}}{\text{total number of pairs}}.$$

Equivalently, tau is the probability of concordance minus the probability of discordance for a pair (x_i, y_i), (x_j, y_j) of observations randomly chosen from the sample. Extending this interpretation to the population leads to the population version of this measure. Analogous to the sample version, we let (X_1, Y_1), (X_2, Y_2) be independent random vectors with a common joint distribution. The population version of Kendall's tau is

$$\tau = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]. \tag{3}$$

6.2 A concordance function

We now generalize the expression for tau given above Let (X_1, Y_1), (X_2, Y_2) be random vectors with (possibly) different joint distribution functions H_1 and H_2, but common marginals F (of X_1 and X_2) and G (of Y_1 and Y_2); and let C_1 and C_2 denote the copulas of (X_1, Y_1) and (X_2, Y_2), respectively. Then $H_1(x, y) = C_1(F(x),G(y))$ and $H_2(x, y) = C_2(F(x), G(y))$. Let K denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2):

$$K = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

We now have the following theorem, which demonstrates that K depends only on the copulas C_1 and C_2 (Nelsen, 1999):

THEOREM 6.2. *Under the conditions above,*

$$K = K(C_1, C_2) = 4 \iint_{\mathbf{I}^2} C_2(u, v) dC_1(u, v) - 1. \tag{4}$$

Some properties of K are as follows:

(i) K is symmetric in its arguments: $K(C_1, C_2) = K(C_2, C_1)$;

(ii) K is nondecreasing in each argument: $C_1(u, v) \leq C_1'(u, v)$ and $C_2(u, v) \leq C_2'(u, v)$ for all (u, v) in $\mathbf{I}^2$ implies $K(C_1, C_2) \leq K(C_1', C_2')$;

(iii) $K(M, M) = 1$, $K(W, W) = -1$, $K(\Pi, \Pi) = 0$, $K(M, \Pi) = 1/3$, $K(W, \Pi) = 1/3$, $K(M, W) = 0$;

(iv) For any C, $K(C, C) \in [-1, 1]$, $K(C, \Pi) \in [-1/3, 1/3]$, $K(C, M) \in [0, 1]$, and $K(C, W) \in [-1, 0]$.

The inequality in (ii) above suggests an ordering $\prec$ of the set $\mathbf{C}$ of copulas:

DEFINITION 6.3. For any pair of copulas C and C', we say that C is *less concordant* than C' (and write $C \prec C'$) whenever $C(u, v) \leq C'(u, v)$ for all (u, v) in $\mathbf{I}^2$.

In Figure 1 we see an illustration of the set $\mathbf{C}$ of copulas partially ordered by $\prec$, and four "concordance axes," each of which, in a sense, locates the position of each copula C within the partially ordered set $(\mathbf{C}, \prec)$.

6.3 Kendall's tau

If X and Y are continuous random variables with copula C, then the population version (3) of Kendall's tau has a succinct expression in terms of K:

$$\tau_{X,Y} = \tau_C = K(C,C) = 4 \iint_{\mathbf{I}^2} C(u,v)dC(u,v) - 1. \tag{5}$$

Thus Kendall's tau is the first "concordance axis" in Figure 1.

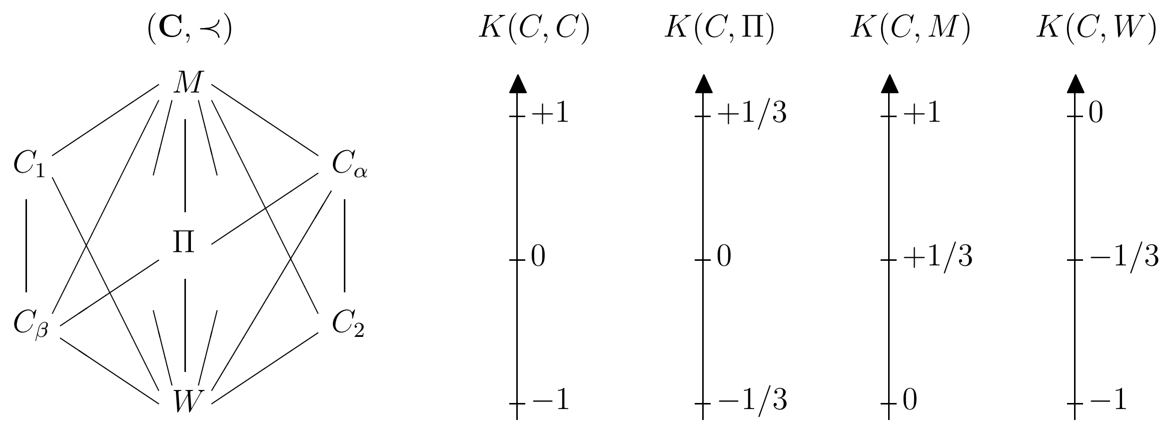

Figure 1: The partial ordered set $(\mathbf{C}, \prec)$ of copulas, and several "concordance axes."

For example, let $C = C_\theta$ be a member of the FGM family: $C_\theta(u,v) = uv + \theta uv(1-u)(1-v), \theta \in [-1,1]$. Then $\tau_C = 2\theta/9$. Since $\tau_C \in [-2/9, 2/9]$, FGM copulas can only model relatively weak dependence.

If C is singular, or possesses a singular component, the form for τ_C given in (5) is not amenable to computation. For many such copulas, the expression

$$\tau_C = 1 - 4 \iint_{\mathbf{I}^2} \frac{\partial}{\partial u} C(u,v) \frac{\partial}{\partial v} C(u,v) du dv \tag{6}$$

is more tractable. It is a consequence of the following theorem (Li et al., 2002):

THEOREM 6.4. Let C_1 *and* C_2 *be copulas. Then*

$$\iint_{\mathbf{I}^2} C_1(u,v)dC_2(u,v) = \frac{1}{2} - \iint_{\mathbf{I}^2} \frac{\partial}{\partial u} C_1(u,v) \frac{\partial}{\partial v} C_2(u,v).$$

For example, let $C_{\alpha,\beta}$ be a member of the Cuadras-Augé family of copulas. When $\alpha, \beta \in (0,1]$, there is a singular component on the curve $u^\alpha = v^\beta$. However, the partial derivatives of $C_{\alpha,\beta}$ are easily evaluated, and as a consequence of (6) we have $\tau_{\alpha,\beta} = \alpha\beta/(\alpha - \alpha\beta + \beta)$.

The integral which appears in (5) can be interpreted as the expected value of the function $C(U,V)$ of uniform $(0,1)$ random variables U and V whose joint distribution function is the copula C:

$$\tau_C = 4E(C(U,V)) - 1 = 4 \int_0^1 t dF_C(t) - 1 = 3 - 4 \int_0^1 F_C(t) dt$$

where F_C denotes the distribution function of the random variable $C(U,V)$.

When C is an Archimedean copula with additive generator ϕ, $F_C(t) = t - \Big(\phi(t)/\phi'(t^+)\Big)$ (Genest and MacKay, 1986ab), and thus

$$\tau_C = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}\,dt.$$

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For example, let C_θ be a Clayton copula with generator ϕ_θ, i.e., $\phi_\theta(t) = \left(t^{-\theta} - 1\right)/\theta$ for $\theta \geq -1$. Then

$$\frac{\phi_\theta(t)}{\phi'_\theta(t)} = \frac{t^{\theta+1} - t}{\theta} \quad (\theta \neq 0), \quad \frac{\phi_0(t)}{\phi'_0(t)} = t \ln t.$$

and hence $\tau_\theta = \theta/\left(\theta + 2\right)$ In the example at the end of Section 4, with $X_{(1)} = \min\left\{X_1, X_2, \cdots, X_n\right\}$ and $X_{(n)} = \max\left\{X_1, X_2, \cdots, X_n\right\}$ for a set of independent identically distributed continuous random variables, the copula of $-X_{(1)}$ and $X_{(n)}$ is a Clayton copula with $\theta = -1/n$, and hence Kendall's tau for$-X_{(1)}$ and $X_{(n)}$ is $-1/(2n-1)$. But $\tau_{X,Y} = -\tau_{-X,Y}$, and thus Kendall's tau for $X_{(1)}$ and $X_{(n)}$ is $1/(2n-1)$.

6.4 Spearman's rho

Let (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) be independent random vectors with a common joint distribution function H (whose margins are F and G), and with copula C. Then the population version of Spearman's rho is defined as the difference between probabilities of concordance and discordance of the vectors (X_1, Y_1) and (X_2, Y_3) (Kruskal, 1958),

$$\rho = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]),$$

(as we shall see below, the coefficient 3 above is a normalization constant). Since the copula of (X_1, Y_1) is C and the copula of (X_2, Y_3) is Π, we have

$$\rho_{X,Y} \;=\; \rho_C \;=\; 3K(C, \Pi).$$

Thus Spearman's rho is essentially the second "concordance axis" in Figure 1, normalized (with the constant 3) so that $\rho_M = +1$ and $\rho_W = -1$.

Evaluating the integral in (4) for ρ_C yields the following expressions:

$$\begin{aligned} \rho_C &= 12\iint_{\mathbf{I}^2} uv\,dC(u,v) - 3, \\ &= 12\iint_{\mathbf{I}^2} C(u,v)\,du\,dv - 3, \\ &= 12\iint_{\mathbf{I}^2} \Big[C(u,v) - uv\Big]\,du\,dv. \end{aligned} \tag{7}$$

The first expression above states that Spearman's rho for continuous random variables X and Y (with distribution functions F and G, respectively, and copula C) is the same as Pearson's product-moment correlation coefficient for the uniform

$(0,1)$ random variables $F(X)$ and $G(Y)$. The second and third expressions yield the following geometric interpretations of ρ_C:

(i) the volume under the graph of $z = C(u,v)$ over $\mathbf{I}^2$ (scaled to lie in $[-1,1]$);

(ii) the signed volume between the graphs of $z = C(u,v)$ and $z = \Pi(u,v)$ (scaled to lie in $[-1,1]$).

6.5 Gini's "coefficient of cograduation"

Early in the last century, Corrado Gini proposed a sample measure of association based on absolute differences in ranks. The population version of that measure, for random variables X and Y with copula C, is given by (Dall'Aglio, 1991, Schweizer, 1991)

$$\gamma = 2 \iint_{\mathbf{I}^2} (|u+v-1| - |u-v|)\, dC(u,v).$$

.

This measure, like Kendall's tau and Spearman's rho, can also be expressed (Nelsen, 1999) in terms of the concordance function K:

$$\gamma_{X,Y} \;=\; \gamma_C \;=\; K(C,M) \;+\; K(C,W).$$

In a sense, Spearman's $\rho_C = 3K(C,\Pi)$ measures a concordance relationship between C and independence (Π), whereas Gini's $\gamma_C = K(C,M) + K(C,W)$ measures a concordance relationship between C and monotone dependence (M and W). Also note that γ_C is equivalent to the sum of the measures on the third and fourth "concordance axes" in Figure 1.

6.6 Quadrant dependence

In discussing dependence properties and measures of association, Kimeldorf and Sampson (1989) noted:"...it is often unclear exactly what dependence (property) a specific measure of association is attempting to describe" We now consider the relationships between the measures discussed above and descriptions of association (other than concordance).

DEFINITION 6.5 (Lehmann, 1966). X and Y are *positively quadrant dependent* $[\mathrm{PQD}(X,Y)]$ if the probability that X and Y are simultaneously "small" is at least as great as it would be were X and Y independent, that is, $\mathrm{PQD}(X,Y)$ if and only if $P(X \le x, Y \le y) \ge P(X \le x)P(Y \le y)$. But this is equivalent to $H(x,y) \ge F(x)G(y)$, which in turn is equivalent to $C(u,v) \ge uv$.

Geometrically, $\mathrm{PQD}(X,Y)$ if and only if the graph of $z = C(u,v)$ lies above the graph of the product (independence) copula $z = \Pi(u,v)$. Negative quadrant dependence (NQD) is defined similarly, and is equivalent to $C(u,v) \le uv$. So the quantity $[C(u,v) - uv]$ measures "local" positive (or negative) quadrant dependence at each point $(u,v) \in \mathbf{I}^2$, and thus $\iint_{\mathbf{I}^2} \Big[C(u,v) - uv\Big]dudv$ is a measure of "average" quadrant dependence. But this integral appears in the third expression in (7), and hence $\rho_C/12$ measures "average" quadrant dependence.

6.7 Likelihood ratio dependence

The result above prompts the following question: Is Kendall's τ_C also an "average" of some dependence property? The answer is yes, and the property is likelihood ratio dependence. It differs from the properties considered above in that it is defined in terms of the joint density function.

DEFINITION 6.6 (Lehmann, 1966). Let X and Y be continuous random variables with joint density function $h(x, y)$. Then X and Y are *positively likelihood ratio dependent* (PLRD(X, Y)) if h satisfies $h(x, y)h(x', y') \geq h(x', y)h(x, y')$ for all $x, x^{'}, y, y^{'}$ in $(-\infty, \infty)$ such that $x \leq x^{'}, y \leq y^{'}$.

The quantity $h(x, y)h(x', y') - h(x', y)h(x, y')$ measures "local" positive (or negative) likelihood ratio dependence at each point $(x, y) \in (-\infty, \infty)^2$, and its integral over the portion of $(-\infty, \infty)^4$ where $x \leq x'$, y $\leq y^{'}$ is a measure of "average" likelihood ratio dependence. In this case, we have (Nelsen, 1992, Nelsen, 1999)

$$\tau_{X,Y} = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y'}\int_{-\infty}^{x'} [h(x, y)h(x', y') - h(x', y)h(x, y')]dxdydx'dy'$$

and hence $\tau_{X,Y}/2$ measures "average" likelihood ratio dependence.

7 The Fréchet-Hoeffding bounds revisited

Again let X, Y be continuous random variables with joint distribution function H, copula C, and marginal distribution functions F and G, respectively. The Fréchet-Hoeffding bounds (1) on H can often be narrowed when we possess additional information about H.

Suppose we know that $H(\tilde{x}, \tilde{y}) = \theta$, where $\tilde{x}$ and $\tilde{y}$ are medians of X and Y, and $\theta \in [0, 1/2]$. Since $F(\tilde{x}) = G(\tilde{y}) = 1/2$, we have $H(\tilde{x}, \tilde{y}) = C(F(\tilde{x}), G(\tilde{y})) = C(1/2, 1/2) = \theta$. Let $\mathbf{C}_\theta$ denote the set of copulas with a common value θ at the point $(1/2, 1/2)$, i.e., $\mathbf{C}_\theta = \{C|\, C \in \mathbf{C},\ C\,(1/2, 1/2) = \theta\}$. If $\underline{C}_\theta$ and $\overline{C}_\theta$ denote, respectively, the pointwise infimum and supremum of $\mathbf{C}_\theta$, i.e., for each (u, v) in $\mathbf{I}^2$, $\underline{C}_\theta(u, v) = \inf\,(C(u, v)|\, C \in \mathbf{C}_\theta)$ and $\overline{C}_\theta(u, v) = \sup\,(C(u, v)|\, C \in \mathbf{C}_\theta)$. $\underline{C}_\theta$ and $\overline{C}_\theta$ are copulas (in fact, shuffles of M—see Section 5) given by

$$\underline{C}_\theta(u, v) = \max\left\{W\,(u, v)\,, \theta - (1/2 - u)^+ - (1/2 - v)^+\right\}$$

and

$$\bar{C}_\theta(u, v) = \min\left\{M\,(u, v)\,, \theta + (u - 1/2)^+ + (v - 1/2)^+\right\}$$

where $x^+ = \max(x, 0)$ (Nelsen, 1999, Nelsen et al., 2001). Thus the best-possible bounds on H are given by

$$\underline{C}_\theta(F(x), G(y)) \leq H(x, y) \leq \overline{C}_\theta(F(x), G(y)).$$

We can apply the same procedure to cases where the additional information about H is the value of a measure of association, that is, we can find best possible copula bounds for sets of copulas such as $\{C|\, C \in \mathbf{C},\ \tau_C = \theta\}$ or $\{C|\, C \in \mathbf{C},\ \rho_C = \theta\}$ (Nelsen et al. 2001).

8 Quasi-copulas

However, it is not always the case that the pointwise best possible upper or lower bound on a set of copulas is a copula. For example, consider $Q(u,v) = \max\{C_1(u,v), C_2(u,v)\}$, where $C_1(u,v) = \min\{u, v, \max(0, u-2/3, v-1/3, u+v-1)\}$, and $C_2(u,v) = C_1(v,u)$. It follows that the Q-volume of the rectangle $[1/3, 2/3]^2$ is $-1/3$. Hence Q is not a copula, however, it is a quasi-copula:

DEFINITION 8.1. A (two-dimensional) *quasi-copula* is a function $Q : \mathbf{I}^2 \to \mathbf{I}$ that satisfies the same boundary conditions (C1) as do copulas, but in place of the 2-increasing condition (C2), the weaker conditions of increasing in each variable (C3) and the Lipschitz condition (C4). Clearly every copula is a quasi-copula, and quasi-copulas which are not copulas are called *proper* quasicopulas.

Conditions C3 and C4 together are equivalent to requiring that the 2-increasing condition $V_Q([a,b]\times[c,d]) = Q(b,d) - Q(a,d) - Q(b,c) + Q(a,c) \geq 0$ holds only when at least one of a,b,c,d is 0 or 1. Geometrically, this means that only those rectangles in $\mathbf{I}^2$ which share a portion of their boundary with $\mathbf{I}^2$ must have nonnegative Q-volume.

Quasi-copulas were introduced in Alsina et al. (1993) (see also Nelsen et al. 1996) in order to characterize operations on univariate distribution functions which can or cannot be derived from corresponding operations on random variables (defined on the same probability space). The original definition was as follows (Alsina et al., 1993):
DEFINITION 8.2. A (two-dimensional) quasi-copula is a function $Q : \mathbf{I}^2 \to \mathbf{I}$ such that for every track B in $\mathbf{I}^2$ (i.e., B can be described as $B = \{(\alpha(t), \beta(t)); 0 \leq t \leq 1\}$ for some continuous and nondecreasing functions α, β with $\alpha(0) = \beta(0) = 0$, $\alpha(1) = \beta(1) = 1$), there exists a copula C_B such that $Q(u,v) = C_B(u,v)$ whenever $(u,v) \in B$.

Genest et al.(1999) established the equivalence of Definitions 8.1 and 8.2, presented the Q-volume interpretation following Definition 8.1, and proved that quasi-copulas also satisfy the Fréchet-Hoeffding bounds inequality (2). For a copula C, the C-volume of a rectangle $R = [a,b]\times[c,d]$ must between 0 and 1 as a consequence of the 2-increasing condition (C2). The next theorem (Nelsen et al., 2002b) presents the corresponding result for quasi-copulas.

THEOREM 8.3. *Let Q be a quasi-copula, and $R = [a,b]\times[c,d]$ any rectangle in $\mathbf{I}^2$. Then $-1/3 \leq V_Q(R) \leq 1$. Furthermore, $V_Q(R) = 1$ if and only if $R = \mathbf{I}^2$, and $V_Q(R) = -1/3$ implies $R = [1/3, 2/3]^2$.*

While Theorem 8.3 limits the Q-volume of a rectangle, the lower bound of $-1/3$ does not hold for more general subsets of $\mathbf{I}^2$. Let μ_Q denotes the finitely additive set function on finite unions of rectangles given by $\mu_Q(S) = \sum_i V_Q(R_i)$ where $S = \bigcup_i R_i$ with $\{R_i\}$ nonoverlapping. Analogous to Theorem 5.1, the copula Π can be approximated arbitrarily closely by quasi-copulas with as much negative "mass" (i.e., value of μ_Q) as desired:

THEOREM 8.4. *Let ϵ, $M > 0$. Then there exists a quasi-copula Q and a set $S \subseteq \mathbf{I}^2$ such that $\mu_Q(S) < -M$* and

$$\sup_{u,v\in\mathbf{I}} |Q(u,v) - \Pi(u,v)| < \varepsilon.$$

.

The proof in (Nelsen et al., 2002b) is constructive, and can be generalized by replacing Π by any quasi-copula whatsoever.

As a consequence of the example at the beginning of this section, the partially ordered set $(\mathbf{C}, \prec)$ is not a lattice, since not every pair of copulas has a supremum and infimum in the set $\mathbf{C}$. However, if we order $\mathbf{Q}$, the set of quasi-copulas, with the same order $\prec$ in Definition 6.3, it can be shown (Nelsen and Úbeda Flores) that $(\mathbf{Q}, \prec)$ is a complete lattice (i.e., every subset of $\mathbf{Q}$ has a supremum and infimum in $\mathbf{Q}$). Furthermore, $(\mathbf{Q}, \prec)$ is order-isomorphic to the Dedekind-MacNeille completion of $(\mathbf{C}, \prec)$. Thus the set of quasi-copulas is a lattice-theoretic completion of the set of copulas, analogous to Dedekind's construction of the reals as a completion by cuts of the set of rationals. Consequently, we have the following characterization of quasi-copulas in terms of copulas.

THEOREM 8.5. *Let $Q : \mathbf{I}^2 \to \mathbf{I}$. Then Q is a quasi-copula if and only if there exists a set $S \neq \emptyset$ of copulas such that for all (u,v) in $\mathbf{I}^2$, $Q(u, v) = \sup\{C(u, v)|C \in S\}$.*

9 Multivariate copulas and quasi-copulas

In this section we extend some of the results in the preceding sections to the multivariate case. While many of the definitions and theorems have analogous multivariate versions, not all do, so we must proceed with care. Some new notation will be advantageous here. We will use vector notation for points in n-dimensional space, e.g., $\mathbf{u} = (u_1, u_2, \cdots, u_n)$; and we will write $\mathbf{a} \leq \mathbf{b}$ when $a_k \leq b_k$ for all k. For $\mathbf{a} \leq \mathbf{b}$, we will let $[\mathbf{a}, \mathbf{b}]$ denote the *n-box* $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, the Cartesian product of n closed intervals. The *vertices* of an n-box are the points $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ where each c_k is equal to either a_k or b_k.

DEFINITION 9.1. An *n-dimensional copula* (or n-copula) is a function $C : \mathbf{I}^n \to \mathbf{I}$ such that:

(i) for every $\mathbf{u}$ in $\mathbf{I}^n$, $C(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $C(\mathbf{u}) = u_k$ if all coor-dinates of $\mathbf{u}$ are 1 except u_k;

(ii) C is *n-increasing*: for every $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{I}^n$ such that $\mathbf{a} \leq \mathbf{b}, V_C([\mathbf{a}, \mathbf{b}]) = \sum(\mathbf{c})C(\mathbf{c}) \geq 0$, where the sum is over the vertices $\mathbf{c}$ of $[\mathbf{a}, \mathbf{b}]$ and $sgn(\mathbf{c}) = 1$ if $c_k = a_k$ for an even number of ks, and -1 if $c_k = a_k$ for an odd number of ks.

DEFINITION 9.2. An *n-dimensional quasi-copula* (or n-quasi-copula) is a function $Q : \mathbf{I}^n \to \mathbf{I}$ such that:

(i) for every $\mathbf{u}$ in $\mathbf{I}^n$, $Q(\mathbf{u}) = 0$ if at least one coordinate of $\mathbf{u}$ is 0, and $Q(\mathbf{u}) = u_k$ if all coordinates of are 1 except u_k;

(ii) Q is nondecreasing in each variable;

(iii) Q satisfies the following Lipschitz condition: for all $\mathbf{u}$ and $\mathbf{u}'$ in $\mathbf{I}^n$,

$$|Q(\mathbf{u}) - Q(\mathbf{u}')| \leq \sum_{i=1}^{n} |u_i - u_i'|.$$

In n dimensions, the Fréchet-Hoeffding inequality (2) is $W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u})$, where $W^n(\mathbf{u}) = \max(u_1 + u_2 + \cdots + u_n - n + 1, 0)$ and $M^n(\mathbf{u}) = \min(u_1, u_2, \cdots, u_n)$.

It has become standard to use superscripts to denote dimension for these Fréchet-Hoeffding bounds. This is a consequence of the fact that they can be constructed iteratively from (the two-dimensional) W and M. The function M^n is an n-copula for all n, however W^n, is a proper n-quasi-copula for all $n \geq 3$ (since $V_{W^n}([\mathbf{1/2}, \mathbf{1}]) = 1 - n/2$). However W^n, is the pointwise best possible lower bound for the set of n-copulas.

In general, constructing n-copulas is difficult. One of the most important open problems is the compatibility problem. For $n = 3$, it is: given three 2-copulas C_1, C_2 and C_3, construct a 3-copula C with C_1, C_2 and C_3 as its 2-dimensional margins, i.e., such that $C(1, v, w) = C_1(v, w), C(u, 1, w) = C_2(u, w)$ and $C(u, v, 1) = C_3(u, v)$. See Joe (1997) and Nelsen (1999) for details.

However, the associativity property enables us to often (but not always) extend Archimedean copulas to higher dimensions. The construction in Section 4 readily extends to n dimensions. Let ϕ be a continuous strictly decreasing function from $\mathbf{I}$ to $[0, \infty]$ such that $\phi(1) = 0$, where $\phi^{[-1]}$ again denotes the "pseudo-inverse" of ϕ. Let Q be the function from $\mathbf{I}^2$ to $\mathbf{I}$ given by

$$Q(u_1, u_2, \cdots, u_n) = \phi^{[-1]}(\phi(u_1) + \phi(u_2) + \cdots + \phi(u_n))$$

Then Q satisfies the boundary conditions for an n-quasi-copula, and is nondecreasing in each variable. The Lipschitz condition is satisfied if and only if ϕ is convex, hence we have

THEOREM 9.3. *Let Q and ϕ be as given above. Then Q is an n-quasi-copula if and only if ϕ is convex.*

Thus Archimedean 2-copulas readily extend to Archimedean n-quasi-copulas. However, there are no proper Archimedean 2-quasi-copulas. since the Lipschitz condition for $n = 2$ is equivalent to the 2-increasing property (Schweizer and Sklar, 1983, Nelsen, 1999). However, for every $n \geq 3$, there are proper Archimedean n-quasi-copulas, for example, W^n.

An Archimedean n-quasi-copula Q will be an n-copula when the following monotonicity condition holds (Kimberling, 1974):

$$(-1)^k \frac{d^k}{dt^k}\phi^{[-1]}(t) \geq 0 \quad for\ all\ t \in (0, \infty) \quad and\ k = 0, 1, \cdots, n$$

The condition is sufficient but not necessary; when it does not hold, Q may an n-copula or a proper n-quasi-copula.

Many properties of Archimedean n-copulas are actually properties of Archimedean n-quasi-copulas. These include

1. $Q(u, u, ..., u) < u$ for every u in $(0, 1)$;
2. if $c > 0$ is any constant, then $c\phi$ is also a generator of Q;
3. If π denotes any permutation of $\{1, 2, \cdots, n\}$, then

$$Q\left(u_{\pi(1)}, u_{\pi(2)}, \cdots, u_{\pi(n)}\right) = Q(u_1, u_2, \cdots, u_n);$$

4. Q is associative in the following sense: if π and π' are any permutations of $\{1, 2, \cdots, 2n-1\}$, then

$$Q\left(u_{\pi(1)}, \cdots, u_{\pi(i-1)}, Q\left(u_{\pi(i)}, \cdots, u_{\pi(i+n-1)}\right), u_{\pi(i+n)}, \cdots, u_{\pi(2n-1)}\right) =$$

$$Q\left(u_{\pi'(1)},\cdots,u_{\pi'(j-1)},Q\left(u_{\pi'(j)},\cdots,u_{\pi'(j+n-1)}\right),u_{\pi'(j+n)},\cdots,u_{\pi'(2n-1)}\right)$$

for all $i,j\in\{1,2,\cdots,n\}$; etc. See Nelsen et al. (2002a) for details.

10 Some open problems

Perhaps the most important open problem concerning copulas is the compatibility problem mentioned in the preceding section. The following four open problems concern Archimedean copulas. The first two are from Alsina et al (2003):

1. There are numerous statistical arguments that are used to justify the assumption of normality. Are there any similar arguments that can be used to justify the assumption that the copula of two random variables is Archimedean?

2. Are there any statistical properties of two random variables which assure that their copula is Archimedean or, more generally, associative?

3. If an Archimedean copula is appropriate for a given data set, are there statistical procedures for choosing a particular family (i.e., for choosing the generator)?

4. It is well-known that for any 2-copula C, the property that $C(u,u)<u$ on $(0,1)$ and associativity characterize the fact that C is Archimedean (Ling, 1965). Is the corresponding statement true for n-quasi-copulas? for n-copulas?

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